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PROXIMAL FLOWS OF LIE GROUPS

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ABSTRACT
Recent results of M. Ratner enable us to solve an open problem on the existence of proximal (and not strongly proximal) minimal actions of Lie groups.

Let (G, X) be a G-flow, i.e. X is compact Hausdorff and G a locally compact group acting on X ($(g, x) \rightarrow gx$ is continuous and $g \rightarrow L_g$ is a homomorphism of G into the group of self-homeomorphisms of X). This action induces an action of G on the space $\mathcal{M}(X)$ of probability measures on X endowed with the weak $*$ topology, making $(G, \mathcal{M}(X))$ a G-flow. We say that $Y \subset X$ is a *minimal* set or that (G, Y) is a *minimal G-flow* if $\overline{Gy} = Y$ for every $y \in Y$. (G, X) is *proximal* if $\Delta = \{(x, x) : x \in X\}$ contains all the minimal subsets of the flow $(G, X \times X)$ where $g(x, y) = (gx, gy)$. (G, X) is *strongly proximal* (s.p.) if $\tilde{X} = \{\delta_x : x \in X\}$, the collection of point masses, contains every minimal subset of $(G, \mathcal{M}(X))$.

For every group G there exists a unique, up to isomorphism, minimal proximal (s.p.) flow $(G, \Pi(G))$ $((G, \Pi_s(G)))$ which admits every minimal proximal (s.p.) G-flow as a factor, [2]. When G is solvable $\Pi_s(G)$ is trivial and $\Pi(G)$ is trivial for nilpotent G. When G is a Lie group it was shown by Furstenberg that $\Pi_s(G)$ is a G-homogeneous space and in particular for G semisimple with finite center $\Pi_s(G) = G/P$ where P is the normalizer of N in G and $G = KAN$ is an Iwasawa decomposition, [3]. The following question was posed in [2]: for a Lie group G is $(G, \Pi(G))$ isomorphic to $(G, \Pi_s(G))$, in other words are there minimal proximal and non-strongly-proximal G-flows. We use recent profound results of M. Ratner on the horocycle flows to show that the action of the solvable group

$$S = \left\{ \begin{pmatrix} a & t \\ 0 & a^{-1} \end{pmatrix} : a, t \in \mathbf{R}, a > 0 \right\}$$

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on the homogeneous space G/Γ where $G = \mathrm{SL}(2, \mathbf{R})$ and Γ is a maximal discrete co-compact and non-arithmetic subgroup, is proximal. Since

$$S \supset N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\}$$

and $(N, G/\Gamma)$, the horocycle flow, is minimal, so is $(S, G/\Gamma)$. Since S is solvable the latter flow can not be strongly proximal. (This can be seen directly since the Haar measure on G/Γ is left invariant by G and hence by S.) The same of course is true for the flow $(G, G/\Gamma)$. Thus we conclude that $\Pi(S)$ is not trivial and that $\Pi(G) \neq \Pi_s(G)$.

Further results of Ratner permit us to conclude also that for every uniform (i.e. discrete and co-compact) subgroup Γ of G, there exists a maximal uniform non-arithmetic subgroup Γ_2 such that $(\Gamma, G/\Gamma_2)$ is minimal and proximal. In particular $\Pi(\Gamma) \neq \Pi_s(\Gamma)$ for every uniform subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$.

We now proceed to show that $(S, G/\Gamma)$ is proximal when Γ is a maximal uniform non-arithmetic subgroup of G. For $s \in \mathbf{R}$ let

$$g_s = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix},$$

then $g_s \in S$ and for

$$h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in N$$

we have $g_s h_t = h_{e^s t} g_s$. Put $X = G/\Gamma$; our first step is to show that every minimal set in $(N, X \times X)$ is of the form $\Delta_t = \{(x, h_t x) : x \in X\}$ for some $t \in \mathbf{R}$. In fact if $M \subset X \times X$ is an N-minimal set then there exists an N-invariant probability measure μ on M, which by unique ergodicity of (N, X), [1], is a self joining. By [4] th. 8.3 and cor. 8.2, there exists $t \in \mathbf{R}$ such that the set Δ_t has μ-measure 1. Since both Δ_t and M are minimal we have $M = \Delta_t$.

Next let $x, y \in X$; there is an N-minimal set contained in $\overline{N(x, y)}$ and by the preceding paragraph there exists $t \in \mathbf{R}$ with $\Delta_t \subset \overline{N(x, y)}$. If $t = 0$ we are done. If not then for $s \in \mathbf{R}$ we have $g_s \Delta_t = \Delta_{e^s t}$ and hence $\overline{S(x, y)} \supset \bigcup_{a>0} \Delta_{at}$. By minimality of (N, X) the latter set is dense in $X \times X$. In particular $\overline{S(x, y)} \supset \Delta = \Delta_0$ and x and y are proximal. This completes the proof.

Remarks. (1) Since we have shown that $(G, G/\Gamma)$ is proximal so is $(H, G/\Gamma)$ for every co-compact subgroup H of G.

(2) If two strictly ergodic N-flows (N, X) and (N, Y) carry invariant measures μ_X and μ_Y respectively, with respect to which they are measure theoretically

disjoint, then they are clearly topologically disjoint. Thus cor. 8.1 of [4] implies that for $G = \mathrm{SL}(2, \mathbf{R})$ and uniform subgroups Γ_1, Γ_2 such that either (i) Γ_1 is arithmetic and Γ_2 is not or (ii) Γ_1 and Γ_2 are not conjugate in G, are maximal and non-arithmetic; $(N, G/\Gamma_1)$ and $(N, G/\Gamma_2)$ are topologically disjoint. This implies of course that $(G, G/\Gamma_1)$ and $(G, G/\Gamma_2)$ are topologically disjoint. By [3] IV.5 prop. 5.1 we have in these cases that $(\Gamma_1, G/\Gamma_2)$ is minimal.

(3) Let Γ_1, Γ_2 be non-conjugate, maximal uniform, and non-arithmetic, let Γ be a subgroup of finite index in Γ_1, then $(\Gamma, G/\Gamma_2)$ is proximal. If $M \subset G/\Gamma_2$ is the unique Γ-minimal set in G/Γ_2 then a finite number of Γ_1 translates of M cover G/Γ_2 and hence M has a non-empty interior. Using the uniqueness of M we see that $M = G/\Gamma_2$ and that $(\Gamma, G/\Gamma_2)$ is minimal proximal.

(4) Since every uniform Γ is contained in a maximal uniform Γ_1 with finite index, we see putting (2) and (3) together that for every uniform Γ there exists some non-arithmetic maximal uniform Γ_2 such that $(\Gamma, G/\Gamma_2)$ is proximal minimal and not strongly proximal. Thus for every such Γ, $\Pi(\Gamma) \neq \Pi_s(\Gamma)$.

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